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On (3,3)-Homogeneous Greechie Orthomodular Posets

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We describe (3,3)-homogeneous orthomodular posets for some cardinality of their sets of atoms. We examine a state space and a set of two-valued states of such logics. Particular homogeneous OMPs with exactly k pure states ($k = 1, \ldots, 7, 10, 11$) have been constructed.

1. INTRODUCTION

Homogeneous orthomodular posets (OMPs) are important (Ovchinnikov, 1999; Sultanbekov, 1992). They can be used in constructing counterexamples or OMPs with certain properties of the state space or the automorphisms group (Navara, 1994; Navara and Rogalewicz, 1988; Navara and Tkadlec, 1991).

Let n, m be natural numbers. An OMP, L, is called (n, m)-*homogeneous* ((n, m)-hom.), if its every atom is contained in n maximal, with respect to inclusion, orhogonal sets of atoms (called *blocks*), and every such set of atoms of L is m-element. The well known concrete logics of the form $\mathcal{L}_q^p = \{X \subset \{1, \ldots, pq\} \mid \text{card}\, X \equiv 0\,(\text{ mod}\, q)\}$ (Ovchinnikov, 1999) are nice examples of homogeneous OMPs. (3,3)-homogeneous logics arise when we consider relational OMPs (Harding, 1996) on a finite set. Orthomodular lattices of the kind were examined in Kohler (1982) and Rogalewicz (1989).

Let L be a finite (n, m)-hom. OMP, A the set of all atoms in L, B the set of all blocks in L, $S = S(L)$ the set of all states on L, and S_2 the set of all two-valued states on L. A state s on L is called *pure* if s is an extreme point of the convex set $S(L)$. It is easy to see that $n \cdot \text{card}\, A = m \cdot \text{card}\, B$.

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Theorem 1.1. *(Ovchinnikov, 1999) Suppose that* $S_2\emptyset = \emptyset$ *and* $f \in S_2$. *Then card* $A = mk$, *card* $B = nk$ *where* k =*card* $(f^{-}(1) \cap A)$.

Let us recall some definitions of the theory of concrete logics (Ovchinnikov, 1999; Ovchinnikov and Sultanbekov, 1998). Let Ω be a set and $\mathcal{P}(\Omega)$ the Boolean algebra of all subsets of Ω. A *concrete logic* (c.l.) on Ω is a subset $\mathcal{E}$ of $\mathcal{P}(\Omega)$ satisfying (1) $\Omega \in \mathcal{E}$; (2) $x \in \mathcal{E} \Rightarrow \Omega\backslash x \in \mathcal{E}$; (3) $x, y \in \mathcal{E}, x \cap y = \emptyset \Rightarrow x \cup y \in \mathcal{E}$.

Denote by $V(\mathcal{E})$ the real vector space of all signed measures on $\mathcal{E}$ and put $\mathcal{E}^\circ = \{\mu \in V(\mathcal{P}(\Omega)) | \forall x \in \mathcal{E}\, (\mu(x) = 0)\}$. $\mathcal{E}$ is called *regular* if every signed measure on $\mathcal{E}$ extends to a signed measure on $\mathcal{P}(\Omega)$.

Theorem 1.2. *(Ovchinnikov, 1999; Ovchinnikov and Sultanbekov, 1998) A concrete logic,* $\mathcal{E}$, *is regular iff* $dim\mathcal{E}^\circ + dim V(\mathcal{E}) = card\,\Omega$.

A set $T \subset S_2(L)$ is called *full* if for all $x, y \in L$ it holds $x \leq y \Leftrightarrow \forall f \in T(f(x) \leq f(y)$. OMP L is isomorphic to a c.l. iff $S_2(L)$ is full. In this case, the sets $x' = \{\mu \in S_2(L) | \mu(x) = 1\}$ form a c.l. $\mathcal{E} = \mathcal{E}(L)$ on $\Omega = S_2(L)$ wich is called a *total representation* of L. We also mention a simple criterion of the fullness of a S_2. The condition equivalent to the fullness of S_2 is as follows: if $x, y \in A$ are not orthogonal, then there exists $s \in S_2(L)$ such that $s(x) = s(y) = 1$.

Let us dwell about (3,3)-hom. finite OMPs, L. Let $l_n = l(P_0, \ldots, P_{n-1}, Q_0, \ldots, Q_{n-1})$ be a loop (Kalmbach, 1983) of order n, where P_i denote the atoms of l_n lying in the vertices of a n-polygon and Q_i are the atoms lying in the middles of sides of the n-polygon. So $\{P_i, Q_i, P_{i+1}\}, i = 0, \ldots, n-1$ (indices modulo n) are all blocks of l_n. It is easy to see that a two-valued state, s, on l_n is well determined by $s(P_0), \ldots, s(P_{n-1})$. We use the following abbreviation: $P_{01}Q_0 = \{P_0, P_1, Q_0\}$.

2. (3,3)-HOMOGENEOUS FINITE OMPS

Let L be a (3,3)-hom. OMP. Obviously card A = card $B \geq 15$ and card $A \neq 16$.

Theorem 2.1. *If card* $A = 15$ *then L is isomorphic to* $\mathcal{L}_2^3$.
2. There exist (3,3)-hom. OMPs L_{17}, L_{27} *with card* $A \in \{17, 27\}$. *For* L_{17} *the set* $S_2 = \emptyset$ *and* $S(L_{17})$ *is isomorphic to a segment* $[0; \frac{2}{3}]$. *For OMP* L_{27} *the set* S_2 *is full and total representation of* L_{27} *is regular.*

Proof:

1. Let card A = 15. Using Greechie diagram for L it is easy to write out all two-valued states of L. So, card $S_2 = 6$ and S_2 is full. The total representation of L is minimal and isomorphic to $\mathcal{L}_2^3$.

2. First we construct L_{17}. Let us consider loop $l_7 = l(P_0, \ldots, P_6, Q_0, \ldots, Q_6)$ and add the atoms R_0, R_1, R_2; $R_i \notin l_7 (i = 0, 1, 2)$. For seven blocks of l_7 we add following 10 blocks:

$$P_{04}R_0,\ Q_{025},\ Q_{03}R_2,\ P_1Q_4R_1,\ Q_{136}, Q_{15}R_0,\ P_{25}R_2,\ Q_{246},\ P_{36}R_1,\ R_{012}.$$

Next we prove that $S(L_{17})$ is isomorphic to a segment $[0; \frac{2}{3}]$. Let $s \in S(L_{17})$. Put $s(P_0) = x, s(Q_0) = y, s(Q_6) = z, s(R_0) = t$, and $s(Q_2) = u$. We show that all values of s are described by x.

1) We have $s(P_1) = 1 - x - y, s(P_6) = 1 - x - z, s(P_4) = 1 - x - t$ and $s(Q_5) = 1 - u - y, s(Q_4) = 1 - u - z$. From the blocks $P_{45}Q_4, P_{56}Q_5$ it follows that $s(P_5) = x + t + u + z - 1$ and $t = y$.
2) Now $s(Q_1) = 1 - s(Q_5) - s(R_0) = u, s(P_2) = x + y - u$ and $s(R_1) = 1 - s(P_1) - s(Q_4) = x + y + z + u - 1$. So $s(R_2) = 1 - s(P_1) - s(Q_4) = 2 - x - 2y - u - z$. From the block $P_{25}R_2$ we get $u = x$.
3) Next $s(R_1) = 1 - s(P_1) - s(Q_4) = 2x + y + z - 1$. From the blocks $P_{36}R_1, Q_{136}$ we calculate $s(P_3) = 1 - x - y, s(Q_3) = 1 - x - z$. Then from the block $P_{34}Q_3$ we have $1 = 3 - 3x - 2y - z$, or $z = 2 - 2y - 3x$.
4) If z from 3) is placed to $s(R_2)$ and $s(Q_3)$ then we get $s(R_2) = x, s(Q_3) = 2x + 2y - 1$. So from the blocks $Q_{03}R_2$ we have $x + y = \frac{2}{3}$.

So the state s has only three values – x, $\frac{2}{3} - x$, and $\frac{1}{3}$, namely:

x—on the atoms P_0, Q_1, Q_2, R_2; $(\frac{2}{3} - x)$—on the atoms P_2, Q_0, Q_6, R_0; $\frac{1}{3}$—on all other remaining atoms.

So $S(L_{17})$ is isomorphic to the segment $[0; \frac{2}{3}]$

The Greechie diagram of the OMP L_{27} is a cube in three-dimensional space with three atoms on each edge, with one atom in center of each side of the cube and with the last atom in the center of cube. The blocks of L_{27} are the lines drawing parallel all axes through the atoms. Thus, Greechie diagram of L_{27} is divided into three layers. Then state $s \in S_2$ is called *type 1 (type 2)* if s equals 1 on main (secondary) diagonal in one of the layers. Then S_2 has six states type 1 and six states type 2. So, card S_2=12. Next $\dim \mathcal{E}^\circ = 3$ and $\dim V(\mathcal{E}) = 9$, where $\mathcal{E}$ is the total representation of L_{27}. By theorem 1.2. $\mathcal{E}$ is regular. □

Theorem 2.2. *There exist (3,3)-hom. OMPs with card* $A \leq 19$ *and with exactly* k *pure states* $(k = 1, 2, \ldots, 7, 10, 11)$.

Proof: Let us denote by $H_k(m)$ a (3,3)-hom. logic with card$A = m$ and k pure states of S. Next, we construct a nine OMPs: $H_1(19)$, $H_2(17)$, $H_3(18)$ $H_4(19)$, $H_5(19)$, $H_6(19)$, $H_7(18)$, $H_{10}(18)$, $H_{11}(19)$.

We enumerate atoms of $H_k(m)$ by natural numbers $1, 2, \ldots, m$ and for a block $\{i, j, n\}$ use abbreviation $i - j - n$. Obviously every such OMP has the following 7 blocks: $B_1, \dots, B_7$: 1–2–3, 1–4–5, 1–6–7, 2–8–9, 2–10–11, 3–12–13, 3–14–15.

1) $H_1(19)$. To $B_1, \dots, B_7$ we add the following 12 blocks:

4–8–12, 4–10–14, 5–9–16, 5–11–17, 6–8–15, 6–13–16,
7–9–18, 7–14–17, 10–16–19, 11–13–18, 12–17–19, 15–18–19.

From the systen of linear equations $s(i) + s(j) + s(n) = 1, \{i, j, n\} \in B$ we found the unique solution $s(i) = \frac{1}{3}(i = 1, \dots, 19)$.

2) $H_2(17)$. Consider L_{17} from Theorem 2.1 as $H_2(17)$. Then S has two pure states ($x = 0, x = \frac{2}{3}$).

3) $H_3(18)$. To $B_1, \dots, B_7$ we add the following 11 blocks:

4–8–12, 4–14–16, 5–10–13, 5–17–18, 6–11–12, 6–16–18,
7–9–17, 7–10–15, 8–15–18, 9–13–16, 11–14–17.

Let $s \in S$. Put $s(17) = x, s(18) = y$. Then s has values: x—on the atoms 2, 4, 6, 13, 15; y—on the atoms 1, 9, 10, 12, 14; $1 - x - y$—on the atoms 3, 5, 7, 8, 11, 16. The state space S is isomorphic to a triangle: $0 \le x \le 1, 0 \le y \le 1, x + y \le 1$. So, S has three pure states: (0,0),(1,0),(0,1).

4) $H_4(19)$. To $B_1, \dots, B_7$ we add the following 12 blocks:

4–8–12, 4–10–14, 5–9–13, 5–11–16, 6–8–15, 6–11–17,
7–9–18, 7–10–19, 12–16–18, 13–17–19, 14–17–18, 15–16–19.

Let $s \in S$. Put $s(6) = x, s(8) = y$. Then s has values: $s(1) = 2y - \frac{1}{3}$, $s(2) = 1 - 2y$, $s(3) = s(12) = s(13) = \frac{1}{3}$, $s(4) = s(5) = \frac{2}{3} - y, s(7) = \frac{4}{3} - x - 2y, s(9) = y$, $s(10) = \frac{2}{3} - x$, $s(11) = x + 2y - \frac{2}{3}, s(14) = s(18) = x + y - \frac{1}{3}, s(15) = s(16) = 1 - x - y$, $s(17) = \frac{5}{3} - 2x - 2y$, $s(19) = 2x + 2y - 1$.

The state space S is isomorphic to a parallelogram: $0 \le x \le \frac{2}{3}, \frac{1}{6} \le y \le \frac{1}{2}, \frac{1}{2} \le x + y \le \frac{5}{6}$. So, S has four pure states:

$$(0, \tfrac{1}{2}),\ (\tfrac{1}{3}, \tfrac{1}{6}),\ (\tfrac{2}{3}, \tfrac{1}{6}),\ (\tfrac{1}{3}, \tfrac{1}{2}).$$

5) $H_5(19)$. To $B_1, \dots, B_7$ we add the following 12 blocks:

4–8–12, 4–10–14, 5–9–15, 5–11–13, 6–13–16, 6–15–17,
7–11–18, 7–12–19, 8–17–18, 9–16–19, 10–17–19, 14–16–18.

Let $s \in S$. Put $s(15) = x, s(19) = y$. Then $s(18) = y, s(6) = \frac{1}{3} - x + y$ and s has values:

x—on the atoms 1, 13; $(\frac{2}{3} - x)$— on the atoms 3, 5;
$(\frac{2}{3} - y)$—on the atoms 7, 16, 17; $\frac{1}{3}$—on all other remaining atoms.

The state space S is isomorphic to a pentagon: $0 \le x \le \frac{2}{3}, 0 \le y \le \frac{2}{3}$, $x - y \le \frac{1}{3}$. So, S has five pure states: $(0, 0)$, $(0, \frac{2}{3})$, $(\frac{1}{3}, 0)$, $(\frac{2}{3}, \frac{1}{3})$, $(\frac{2}{3}, \frac{2}{3})$.

6) $H_6(19)$. To $B_1, \dots, B_7$ we add the following 12 blocks:

4–8–12, 4–10–14, 5–9–13, 5–11–16, 6–9–15, 6–11–17,
7–10–18, 7–13–19, 8–16–19, 12–17–18, 14–17–19, 15–16–18.

Let $s \in S$. Put $s(2) = x, s(10) = y$. Then $s(5) = x, s(6) = x - y + \frac{1}{3}, s(11) = 1 - x - y, s(17) = 2y - \frac{1}{3}, s(18) = 1 - 2y$ and s has also values:

y—on the atoms 7, 15, 16; $(\frac{2}{3} - x)$—on the atoms 1, 9;
$(\frac{2}{3} - y)$—on the atoms 14, 19; $\frac{1}{3}$—on the atoms 3, 4, 8, 12, 13.

The state space S is isomorphic to a hexagon: $0 \le x \le \frac{2}{3}, \frac{1}{6} \le y \le \frac{1}{2}, y + x \le 1, y - x \le \frac{1}{3}$. So, S has six pure states:

$(0, \frac{1}{6}),\ (0, \frac{1}{3}),\ (\frac{1}{6}, \frac{1}{2}),\ (\frac{1}{2}, \frac{1}{2}),\ (\frac{2}{3}, \frac{1}{3}),\ (\frac{2}{3}, \frac{1}{6})$.

7) $H_7(18)$. To $B_1, \ldots, B_7$ we add the following 11 blocks:

4–8–12, 4–10–14 5–11–15, 5–16–17, 6–8-18, 6–10–16,
7–9–15, 7–12–17, 9–13–16, 11–13–18, 14–17–18.

Let $s \in S$. Put $s(1) = x, s(3) = y, s(18) = z$. Then $s(2) = 1 - x - y, s(17) = 1 - 2z$ and s has also values: x—on the atoms 8, 10; y—on the atoms 9, 11; $(1 - x - z)$—on the atoms 4, 6; z—on the atoms 5, 7, 12, 14, 16; $(1 - y - z)$—on the atoms 13, 15.

The state space S is isomorphic to a polytope in three-dimensional space: $0 \le x \le 1, 0 \le y \le 1, 0 \le z \le \frac{1}{2}, x + y \le 1, x + z \le 1, y + z \le 1$. So, S has seven pure states:

$(0,0,0),\ (0, 1, 0), (1, 0, 0),\ (0, 0, \frac{1}{2}),\ (0, \frac{1}{2}, \frac{1}{2}),\ (\frac{1}{2}, 0, \frac{1}{2}),\ (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$.

8) $H_{10}(18)$. To $B_1, \ldots, B_7$ we add the following 11 blocks:

4–8–12, 4–10–14 5–8–16, 5–11–17, 6–12–16, 6–14–17,
7–9–13, 7–11–15, 9–17–18, 10–13–18, 15–16–18.

Let $s \in S$. Put $s(6) = x, s(16) = y, s(17) = z$. Then $s(1) = 2y - x, s(11) = 2y - z, s(12) = 1 - x - y, s(14) = 1 - x - z, s(11) = 1 - y - z$ and s has also values:

x—on the atoms 3, 4; y—on the atoms 8, 9, 13;$(1 - 2y)$—on the atoms 2, 5, 7;
z—on the atoms 5, 10; $(1 - y - z)$—on the atoms 13, 15.

The state space S is isomorphic to a polytope in three-dimensional space: $0 \le x \le \frac{2}{3}, 0 \le y \le \frac{1}{2}, 0 \le z \le \frac{2}{3}, x + y \le 1, x + z \le 1, y + z \le 1, \frac{x}{2} \le y, \frac{z}{2} \le y$.

So, S has 10 pure states:

$(0, 0, 0),\ (0, \frac{1}{2}, 0),\ (\frac{1}{2}, \frac{1}{2}, 0),\ (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}),\ (0, \frac{1}{2}, \frac{1}{2}),$
$(0, \frac{1}{3}, \frac{2}{3}),\ (\frac{1}{3}, \frac{1}{3}, \frac{2}{3}),\ (\frac{1}{2}, \frac{1}{4}, \frac{1}{2}),\ (\frac{2}{3}, \frac{1}{3}, \frac{1}{3}),\ (\frac{2}{3}, \frac{1}{3}, 0)$.

9) $H_{11}(19)$. To $B_1, \ldots, B_7$ we add the following 12 blocks:

4–8–12, 4–10–14 5–9–15, 5–11–13, 6–8–16, 6–10–17,
7–13–19, 7–15–18, 9–17–19, 11–16–18, 12–17–18, 14–16–19.

Let $s \in S$. Put $s(10) = x, s(15) = y, s(19) = z$. Then $s(1) = x + y - \frac{1}{3}, s(6) = \frac{1}{3} - x + z, s(7) = 1 - y - z$ and s has also values:

x—on the atom 8; y—on the atom 13; z—on the atom 18;

$\frac{2}{3} - x$—on the atoms 2, 4; $\frac{2}{3} - y$—on the atoms 3, 5; $\frac{2}{3} - z$ —on the atoms 16, 17; $\frac{2}{3} - z$—on the atoms 16, 17; $\frac{1}{3}$—on the atoms 9, 11, 12, 14.

The state space S is isomorphic to a polytope in three-dimensional space: $0 \le x \le \frac{2}{3}, 0 \le y \le \frac{2}{3}, 0 \le z \le \frac{2}{3}, x + y \ge \frac{1}{3}, z \ge x - \frac{1}{3}1, y + z \le 1$.

So, S has 11 pure states:

$$(\tfrac{1}{3}, 0, 0),\ (0, \tfrac{1}{3}, 0),\ (0, \tfrac{2}{3}, 0),\ (\tfrac{1}{3}, \tfrac{2}{3}, 0),\ (\tfrac{2}{3}, \tfrac{2}{3}, \tfrac{1}{3}),\ (\tfrac{2}{3}, 0, \tfrac{1}{3}),$$
$$(0, \tfrac{2}{3}, \tfrac{1}{3}),\ (\tfrac{1}{3}, 0, \tfrac{2}{3}),\ (\tfrac{2}{3}, 0, \tfrac{2}{3}),\ (\tfrac{2}{3}, \tfrac{1}{3}, \tfrac{2}{3}),\ (0, \tfrac{1}{3}, \tfrac{2}{3}).$$

□

Remark 2.3. The special interest have (3,3)-hom. OMPs with a unique state. What least number of atoms of such logic? An example with 22 atoms till now was known (Greechie and Miller, 1970). The example, constructed by us, has 19 atoms, and we did not manage to construct OMP with smaller number of atoms. Probably number of atoms 19 cannot be reduced; we yet have no proof it. In Ptak (1987) was developed a method of construction of OMPs with a unique state, however these logics are not homogeneous.

Using Greechie loop lemma (Kalmbach, 1983) it is not difficult to show, that all listed above (3,3)-hom. logic $H_k(m)$ are not orthomodular lattices. Certainly the examples of orthomodular lattices with such property will have the much greater number of atoms.

Remark 2.4. There is a well-known method of constructing the finite (3,3)-hom. OMPs with even card A. Let $n \ge 9$ and $A = \{a_i | i = 0, 1, \ldots, 2n - 1\}$ be a set of atoms. Then sets $\{a_{2i}, a_{2i+1}, a_{2i+2}\}$; $\{a_{2i-5}, a_{2i}, a_{2i+5}\}$ (indices modulo $2n$) as the blocks generate some (3,3)-hom. logic $L(2n)$. For example, $L(22)$ has one, $L(20)$ has two, and $L(18)$ has three pure states for the corresponding state spaces. But the author is not familiar with any convenient method of constructing such OMPs with odd card A.

Remark 2.5. (3,3)-hom. logics arise when we consider a relational OMPs (Harding, 1996) on a finite set. For example, the relational OMP on 8-element set is (3,3)-homogeneous. Every horizontal summand of this OMP has 28 atoms and exactly one pure state. We present all blocks of this summand:

1–13–15, 1–6–21, 1–14–22, 2–6–25, 2–5–17, 2–18–26, 3–9–13,
3–10–25, 3–14–26, 4–9–17, 4–10–21, 4–18–22, 5–15–19, 6–23–27,
7–11–13, 7–12–25, 7–15–27, 8–11–21, 8–12–17, 8–19–23, 9–16–20,
10–24–28, 11–16–24, 12–20–28, 14–24–27, 15–20–26, 16–19–22, 18–23–28.

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